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ON BICOMPLEX $(k, l)-$GADOVAN NUMBERS

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Abstract. In this study, we introduce the bicomplex $(k, l)-$Gadovan numbers, which can be considered refinements of the Gadovan numbers. We give the recurrence relation, the Binet formula, the generating functions, the exponential generating function of the bicomplex $(k, l)-$Gadovan numbers. Also, we derive Cassini identity, Catalan identity, Vajda identity, Honsberger identity and d'Ocagne identity which are important identities of the bicomplex $(k, l)-$Gadovan numbers for sequences of numbers.

1. Introduction

The bicomplex numbers are an extension of complex numbers and they are defined using two imaginary units. Bicomplex numbers were first described by Corrado Segre in 1892.The set of bicomplex numbers is defined as:

$$\mathbb{BC} = \{z_1 + z_2e_2 \mid z_1, z_2 \in \mathbb{C}, {e_2}^2 = -1\}$$

where z_1, z_2 are complex numbers. Another way to represent bicomplex numbers is:

$$z = z_1 + z_2e_2 = x + ye_1 + (t + ke_1)e_2 = x + ye_1 + te_2 + ke_1e_2$$

where $x, y, t, k \in \mathbb{R}$. Here e_1, e_1 and e_1e_2 are imaginary units and satisfy ${e_1}^2 = {e_2}^2 = -1$, $e_1e_2 = e_2e_1$ the rules [17].Bicomplex numbers are actually complex numbers with complex coefficients [10, 11].Bicomplex numbers have conjugates depending on three imaginary units and have norms depending on these conjugates. These are also shown in Table 1.

Conjugate	Norm
$z_{e_1} = x - ye_1 + (t - ke_1)e_2$	$\lvert z\rvert_{e_1} = \sqrt{z \times z_{e_1}}$
$z_{e_2} = x + ye_1 - (t + ke_1)e_2$	$\lvert z\rvert_{e_2} = \sqrt{z \times z_{e_2}}$
$z_{e_1e_2} = x - ye_1 - (t - ke_1)e_2$	$\lvert z\rvert_{e_1e_2} = \sqrt{z \times z_{e_1e_2}}$

Table 1. Conjugates and Norm

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Bicomplex numbers have been studied by many researchers and their applications have been made with different number sequences [2, 8, 10, 11, 13, 17, 19].

The sequence of the Fibonacci numbers is one of the most notable and famous number sequences in literature. It has found application not only in the field of mathematics but also in many other branches of science. Many researchers have studied Fibonacci numbers and studied their different generalizations [4, 5, 7, 9, 14, 15, 18]. One of these generalizations is the Gadovan numbers. Gadovan Numbers owe their origin to Diskaya and Menken [3]. Gadovan numbers are essentially generalizations of the second order Padovan numbers [1].

Definition 1.1. The Gadovan numbers are defined by the the 3rd order recurrence relation

$$GP_{n+3} = GP_{n+1} + GP_n, \quad n \geq 0$$

with $GP_0 = a$, $GP_1 = b$ and $GP_2 = c$ [3].

Later, a generalization of Gadovan numbers is defined by Özkan et al.[16].

Definition 1.2. $(k,l)-$Gadovan numbers $GP_{(k,l),n}$ are given by the following third order recurrence relation

$$GP_{(k,l),n+3} = kGP_{(k,l),n+1} + lGP_{(k,l),n}, \qquad n \geq 0$$

with initial conditions $GP_{(k,l),0} = a$, $GP_{(k,l),1} = b$ and $GP_{(k,l),2} = c$ [16].

In this work, we define a new generalization of the bicomplex numbers by taking the real components of the $(k,l)-$Gadovan numbers as numbers. We give the Binet formula, the generating functions, the exponential generating function of the bicomplex $(k,l)-$Gadovan numbers. In addition, we derive Cassini identity, Catalan identity, Vajda identity and d'Ocagne identity for these numbers.

2. Bicomplex $(k,l)-$Gadovan Numbers

In this section, we define bicomplex $(k,l)-$Gadovan numbers. We give certain identities regarding these numbers.

Definition 2.1. We define the bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ by

$$BGP_{(k,l),n} = GP_{(k,l),n} + GP_{(k,l),n+1}e_1 + GP_{(k,l),n+2}e_2 + GP_{(k,l),n+3}e_1e_2$$

where $GP_{(k,l),n}$ is nth $(k,l)-$Gadovan numbers and e_1, e_2 and e_1e_2 are bicomplex units.

We show some terms of the bicomplex $(k,l)-$Gadovan numbers in Table 2.

$BGP_{(k,l),0}$	$a + be_1 + ce_2 + (kb + la)e_1e_2$
$BGP_{(k,l),1}$	$b + ce_1 + (kb + la)e_2 + (kc + lb)e_1e_2$
$BGP_{(k,l),2}$	$c + (kb + la)e_1 + (kc + lb)e_2 + (k^2b + kla + lc)e_1e_2$
$BGP_{(k,l),3}$	$kb + la + (kc + lb)e_1 + (k^2b + kla + lc)e_2 + (k^2c + 2klb + l^2a)e_1e_2$

Table 2. Some terms of bicomplex $(k,l)-$Gadovan numbers

Definition 2.2. The real and imaginary parts bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ respectively, as follows.

$i)$ $Re(BGP_{(k,l),n)}) = GP_{(k,l),n}$,

$ii)$ $Im(BGP_{(k,l),n)}) = GP_{(k,l),n+1}e_1 + GP_{(k,l),n+2}e_2 + GP_{(k,l),n+3}e_1e_2$.

Definition 2.3. The conjugate of bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ is defined by

$$\begin{aligned}
i)\ \overline{(BGP_{(k,l),n})}_{e_1} &= GP_{(k,l),n} - GP_{(k,l),n+1}e_1 + GP_{(k,l),n+2}e_2 - GP_{(k,l),n+3}e_1e_2,\\
ii)\ \overline{(BGP_{(k,l),n})}_{e_2} &= GP_{(k,l),n} + GP_{(k,l),n+1}e_1 - GP_{(k,l),n+2}e_2 - GP_{(k,l),n+3}e_1e_2,\\
iii)\ \overline{(BGP_{(k,l),n})}_{e_1e_2} &= GP_{(k,l),n} - GP_{(k,l),n+1}e_1 - GP_{(k,l),n+2}e_2 + GP_{(k,l),n+3}e_1e_2.
\end{aligned}$$

Definition 2.4. The norm of bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ is defined by

$$\begin{aligned}
i)\ \|(BGP_{(k,l),n})_{e_1}\|^2 &= GP^2_{(k,l),n} + GP^2_{(k,l),n+1} + GP^2_{(k,l),n+2} - GP^2_{(k,l),n+3}\\
&\quad 2(GP_{(k,l),n+1}GP_{(k,l),n+3} + GP_{(k,l),n}GP_{(k,l),n+2})e_2,\\
ii)\ \|(BGP_{(k,l),n})_{e_2}\|^2 &= GP^2_{(k,l),n} - GP^2_{(k,l),n+1} + GP^2_{(k,l),n+2} - GP^2_{(k,l),n+3}\\
&\quad 2(GP_{(k,l),n}GP_{(k,l),n+1} + GP_{(k,l),n+2}GP_{(k,l),n+3})e_1,\\
iii)\ \|(BGP_{(k,l),n})_{e_1e_2}\|^2 &= GP^2_{(k,l),n} + GP^2_{(k,l),n+1} + GP^2_{(k,l),n+2} + GP^2_{(k,l),n+3}\\
&\quad 2(GP_{(k,l),n}GP_{(k,l),n+3} - GP_{(k,l),n+1}GP_{(k,l),n+2})e_1e_2.
\end{aligned}$$

Definition 2.5. The matrix representation of the bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ is as follows.

$$M_n = \begin{bmatrix} GP_{(k,l),n} & -GP_{(k,l),n+1} & -GP_{(k,l),n+2} & GP_{(k,l),n+3} \\ GP_{(k,l),n+1} & GP_{(k,l),n} & -GP_{(k,l),n+3} & -GP_{(k,l),n+2} \\ GP_{(k,l),n+2} & -GP_{(k,l),n+3} & GP_{(k,l),n} & -GP_{(k,l),n+1} \\ GP_{(k,l),n+3} & GP_{(k,l),n+2} & GP_{(k,l),n+1} & GP_{(k,l),n} \end{bmatrix}$$

Theorem 2.1. *Bicomplex $(k,l)-$Gadovan numbers $BGP_{(kl),n}$ satisfy the following equation.*

$$BGP_{(k,l),n+3} = kBGP_{(k,l),n+1} + lBGP_{(k,l),n}$$

Proof. $BGP_{(k,l),n+3}$

$$\begin{aligned}
&= GP_{(k,l),n+3} + GP_{(k,l),n+4}e_1 + GP_{(k,l),n+5}e_2 + GP_{(k,l),n+6}e_1e_2\\
&= (kGP_{(k,l),n+1} + lGP_{(k,l),n}) + (kGP_{(k,l),n+2} + lGP_{(k,l),n+1})e_1\\
&+ (kGP_{(k,l),n+3} + lGP_{(k,l),n+2})e_2 + (kGP_{(k,l),n+4} + lGP_{(k,l),n+3})e_1e_2\\
&k(GP_{(k,l),n+1} + GP_{(k,l),n+2}e_1 + GP_{(k,l),n+3}e_2 + GP_{(k,l),n+4}e_1e_2)\\
&+ l(GP_{(k,l),n} + GP_{(k,l),n+1}e_1 + GP_{(k,l),n+2}e_2 + GP_{(k,l),n+3}e_1e_2)\\
&= kBGP_{(k,l),n+1} + lBGP_{(k,l),n}.
\end{aligned}$$

□

Theorem 2.2. *Binet Formula for bicomplex $(k,l)-$Gadovan numbers $BGP_{(k,l),n}$ is*

$$BGP_{(k,l),n} = Ax_1^n \widehat{x}_1 + Bx_2^n \widehat{x}_2 + Cx_3^n \widehat{x}_3$$

where

$$A = \frac{c - bx_2 - bx_3 + ax_2x_3}{x_1^3 - x_1^2x_2 - x_1^2x_3 + x_1x_2x_3},$$
$$B = \frac{c - bx_1 - bx_3 - ax_1x_3}{x_2^3 - x_2^2x_3 - x_2^2x_1 + x_1x_2x_3},$$
$$C = \frac{c - b.x_1 - b.x_2 - a.x_1.x_2}{x_3^3 - x_1x_3^2 - x_3^2x_2 + x_1x_2x_3},$$
$$\widehat{x}_1 = 1 + x_1e_1 + x_1^2e_2 + x_1^3e_1e_2,$$
$$\widehat{x}_2 = 1 + x_2e_1 + x_2^2e_2 + x_2^3e_1e_2,$$
$$\widehat{x}_3 = 1 + x_3e_1 + x_3^2e_2 + x_3^3e_1e_2.$$

Proof. From

$$GP_{k,n} = Ax_1^n + Bx_2^n + Cx_3^n,$$

we have

$$\begin{aligned} BGP_{(k,l),n} &= GP_{(k,l),n} + GP_{(k,l),n+1}e_1 +, GP_{(k,l),n+2}e_2 + GP_{(k,l),n+3}e_1e_2 \\ &= Ax_1^n + Bx_2^n + Cx_3^n + (Ax_1^n + Bx_2^n + Cx_3^n)e_1 \\ &\ + (Ax_1^n + Bx_2^n + Cx_3^n)e_2 + (Ax_1^n + Bx_2^n + Cx_3^n)e_1e_2 \\ &= Ax_1^n(1 + x_1e_1 + x_1^2e_2 + x_1^3e_1e_2) + Bx_2^n(1 + x_2e_1 + x_2^2e_2 + x_2^3e_1e_2) \\ &\ + Cx_3^n(1 + x_3e_1 + x_3^2e_2 + x_3^3e_1e_2) \\ &= Ax_1^n \widehat{x}_1 + Bx_2^n \widehat{x}_2 + Cx_3^n \widehat{x}_3. \end{aligned}$$

Thus, the desired result is achieved. □

Theorem 2.3. *The generating function of $BGP_{(k,l),n}$ is*

$$\sum_{n=1}^{\infty} BGP_{(k,l)n}x^n = \frac{BGP_{(k,l),0} + BGP_{(k,l),1}x + x^2(BGP_{(k,l),2} - BGP_{(k,l),0})}{1 - kx^2 - l\ x^3}.$$

Proof. Let

$$G(x) = \sum_{n=1}^{\infty} BGP_{(k,l)n}x^n.$$

$$G(x) = BGP_{(k,l),0} + BGP_{(k,l),1}x + BGP_{(k,l),2}x^2 + BGP_{(k,l),3}x^3 + \cdots + BGP_{(k,l),n}x^n + \ldots \tag{2.1}$$

respectively multiplying both sides of the identity (2.1) by $-kx^2$ and $-lx^3$.

$$\begin{aligned} -kx^2G(x) &= -BGP_{(k,l),0}kx^2 + BGP_{(k,l),1}kx^3 - BGP_{(k,l),2}kx^4 - BGP_{(k,l),3}kx^5 \\ &\quad - \cdots - BGP_{(k,l),n}kx^{n+2} - \ldots \end{aligned} \tag{2.2}$$

$$-lx^3G(x) = -BGP_{(k,l),0}lx^3 - BGP_{(k,l),1}lx^4 - BGP_{(k,l),2}lx^5 - BGP_{(k,l),3}lx^6 - \cdots - BGP_{(k,l),n}lx^{n+3} - \ldots \tag{2.3}$$

From (2.1), (2.2) and (2.3), we get

$$G(x)\left(1-kx^2-l\,x^3\right) = BGP_{(k,l),0} + BGP_{(k,l),1}x + BGP_{(k,l),2}x^2 - BGP_{(k,l),0}kx^2$$

$$G(x) = \frac{BGP_{(k,l),0} + BGP_{(k,l),1}x + x^2(BGP_{(k,l),2} - BGP_{(k,l),0})}{1-kx^2-lx^3}.$$

□

Theorem 2.4. *The exponential generating function for bicomplex $(k,l)-$Gadovan numbers*

$$\sum_{n=0}^{\infty}\frac{BGP_{(k,l),n}x^n}{n!} = A\widehat{x}_1e^{x_1x} + B\widehat{x}_2e^{x_2x} + C\widehat{x}_3x_3x.$$

Proof. For the proof, we use the Binet formula of bicomplex $(k,l)-$Gadovan numbers.

$$\sum_{n=0}^{\infty}\frac{BGP_{(k,l),n}x^n}{n!} = \sum_{n=0}^{\infty}\frac{(Ax_1^n\widehat{x}_1 + Bx_2^n\widehat{x}_2 + Cx_3^n\widehat{x}_3)x^n}{n!}$$
$$= A\widehat{x}_1\sum_{n=0}^{\infty}\frac{(x_1x)^n}{n!} + B\widehat{x}_2\sum_{n=0}^{\infty}\frac{(x_2x)^n}{n!} + C\widehat{x}_3\sum_{n=0}^{\infty}\frac{(x_3x)^n}{n!}.$$

We know that

$$e^{x_1x} = \sum_{n=1}^{\infty}\frac{(x_1x)^n}{n!},$$

$$e^{x_2x} = \sum_{n=1}^{\infty}\frac{(x_2x)^n}{n!},$$

and

$$e^{x_3x} = \sum_{n=1}^{\infty}\frac{(x_3x)^n}{n!}.$$

So,

$$\sum_{n=0}^{\infty}\frac{BGP_{(k,l),n}x^n}{n!} = A\widehat{x}_1e^{x_1x} + B\widehat{x}_2e^{x_2x} + C\widehat{x}_3x_3x.$$

Thus, the proof is completed. □

Theorem 2.5. *For $m,n \in \mathbb{Z}^+$, there is the following equation.*

$$\sum_{n=1}^{m=1}\binom{m}{n}k^nl^{m-n}BGP_{(k,l),n} = BGP_{(k,l),3m}.$$

Proof.

$$\sum_{n=1}^{m}\left(\begin{array}{c} m \\ n \end{array}\right) k^{n} l^{m-n} BGP_{(k,l),n} =$$

$$\sum_{n=1}^{m}\left(\begin{array}{c} m \\ n \end{array}\right)\left(A\widehat{x}_1(kx_1)^n + B\widehat{x}_2(kx_2)^n + C\widehat{x}_3(kx_3)^n\right) l^{m-n} =$$

$$A\widehat{x}_1 \sum_{n=1}^{m}\left(\begin{array}{c} m \\ n \end{array}\right)(kx_1)^n l^{m-n} + B\widehat{x}_2 \sum_{n=1}^{m}\left(\begin{array}{c} m \\ n \end{array}\right)(kx_2)^n l^{m-n} +$$

$$C\widehat{x}_2 \sum_{n=1}^{m}\left(\begin{array}{c} m \\ n \end{array}\right)(kx_3)^n l^{m-n} =$$

$$A\widehat{x}_1(kx_1+l)^m + B\widehat{x}_2(kx_2+l)^m + C\widehat{x}_3(kx_3+l)^m =$$

$$A\widehat{x}_1 {x_1}^{3m} + B\widehat{x}_2 {x_2}^{3m} + C\widehat{x}_3 {x_3}^{3m} = BGP_{(k,l),3m}.$$

□

Theorem 2.6. *For $m, n \in \mathbb{Z}^{+}$, there is the following equation.*

$$\sum_{r=1}^{m}\binom{m}{r} k^{m-r} l^{m} BGP_{(k,l),n-r} = BGP_{(k,l),n+2m}.$$

Proof.

$$\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right) k^{m-r} l^{m} BGP_{(k,l),n-r} =$$

$$\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right) k^{m-r}\left(Ax_1^{n-r} + Bx_2^{n-r} + Cx_3^{n-r}\right) l^{m}$$

$$= A\widehat{x}_1\left[\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right) k^{m-r} x_1^{m-r} l^{m}\right] x_1^{n-m} + B\widehat{x}_2\left[\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right) k^{m-r} x_2^{m-r} l^{m}\right] x_2^{n-m}$$

$$+ C\widehat{x}_3\left[\sum_{r=1}^{m=1}\left(\begin{array}{c} m \\ r \end{array}\right) k^{m-r} x_3^{m-r} l^{m}\right] x_3^{n-m}$$

$$= A\widehat{x}_1\left[\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right)(kx_1)^{m-r} l^{r}\right] x_1^{n-m} + B\widehat{x}_2\left[\sum_{r=1}^{m}\left(\begin{array}{c} m \\ r \end{array}\right)(kx_2)^{m-r} l^{r}\right] x_2^{n-m}$$

$$+ C\widehat{x}_3\left[\sum_{r=1}^{m=1}\left(\begin{array}{c} m \\ r \end{array}\right)(kx_3)^{m-r} l^{r}\right] x_3^{n-m}$$

$$= A\widehat{x}_1(kx_1+l)^m x_1^{n-m} + B\widehat{x}_2(kx_2+l)^m x_2^{n-m} + C\widehat{x}_3(kx_3+l)^m x_3^{n-m}$$

$$= A\widehat{x}_1 {x_1}^{n+2m} + B\widehat{x}_2 {x_2}^{n+2m} + C\widehat{x}_3 {x_3}^{n+2m} = BGP_{(k,l),n+2m}.$$

Thus, the proof is obtained. □

Theorem 2.7. *(Cassini Identity) For $n \geq 1$, we have*

$$\begin{aligned}
&BGP_{(k,l),n-1}BGP_{(k,l),n+1} - {BGP_{(k,l),n}}^2 = \\
&l^{n-1}\left(AB{x_3}^{1-n}\widehat{x}_1\widehat{x}_2(x_1-x_2)^2 + AC{x_2}^{1-n}\widehat{x}_1\widehat{x}_3(x_1-x_3)^2\right. \\
&\quad \left. +BC{x_1}^{1-n}\widehat{x}_2\widehat{x}_3(x_2-x_3)^2\right).
\end{aligned}$$

Proof. We use the Binet formula of the bicomplex $(k,l)-$Gadovan numbers for the proof.

$$\begin{aligned}
&BGP_{(k,l),n-1}BGP_{(k,l),n+1} - {BGP_{(k,l),n}}^2 \\
&= \left(Ax_1^{n-1}\widehat{x}_1 + Bx_2^{n-1}\widehat{x}_2 + Cx_3^{n-1}\widehat{x}_3\right)\left(Ax_1^{n+1}\widehat{x}_1 + Bx_2^{n+1}\widehat{x}_2 + Cx_3^{n+1}\widehat{x}_3\right) \\
&\quad - \left(Ax_1^{n}\widehat{x}_1 + Bx_2^{n}\widehat{x}_2 + Cx_3^{n}\widehat{x}_3\right)^2 \\
&= ABx_1^n x_2^n\widehat{x}_1\widehat{x}_2\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}-2\right) + ACx_1^n x_3^n\widehat{x}_1\widehat{x}_3\left(\frac{x_1}{x_3}+\frac{x_3}{x_1}-2\right) \\
&\quad + BCx_2^n x_3^n\widehat{x}_2\widehat{x}_3\left(\frac{x_2}{x_3}+\frac{x_3}{x_2}-2\right) \\
&= AB(x_1x_2)^{n-1}\widehat{x}_1\widehat{x}_2(x_1-x_2)^2 + AC(x_1x_3)^{n-1}\widehat{x}_1\widehat{x}_3(x_1-x_3)^2 \\
&\quad + BC(x_2x_3)^{n-1}\widehat{x}_2\widehat{x}_3(x_2-x_3)^2 \\
&= ABl^{n-1}{x_3}^{1-n}\widehat{x}_1\widehat{x}_2(x_1-x_2)^2 + ACl^{n-1}{x_2}^{1-n}\widehat{x}_1\widehat{x}_3(x_1-x_3)^2 \\
&\quad + BCl^{n-1}{x_1}^{1-n}\widehat{x}_2\widehat{x}_3(x_2-x_3)^2 \\
&= l^{n-1}\left(AB{x_3}^{1-n}\widehat{x}_1\widehat{x}_2(x_1-x_2)^2 + AC{x_2}^{1-n}\widehat{x}_1\widehat{x}_3(x_1-x_3)^2\right. \\
&\quad \left. +BC{x_1}^{1-n}\widehat{x}_2\widehat{x}_3(x_2-x_3)^2\right).
\end{aligned}$$

So, the desired is obtained. □

Theorem 2.8. *(Catalan Identity) For $n \geq t$, we have*

$$\begin{aligned}
&BGP_{(k,l),n-t}BGP_{(k,l),n+t} - {BGP_{(k,l),n}}^2 = \\
&k^{n-1}\left(AB{x_3}^{n-1}(x_1-x_2)^2 + AC{x_2}^{n-1}(x_1-x_3)^2 \ + BC{x_1}^{n-1}(x_2-x_3)^2\right).
\end{aligned}$$

Proof. We use the Binet formula of the bicomplex $(k,l)-$Gadovan numbers for the proof.

$$
\begin{aligned}
& BGP_{(k,l),n-t} BGP_{(k,l),n+t} - {BGP_{(k,l),n}}^{2} \\
& = \left(Ax_1^{n-t}\widehat{x}_1 + Bx_2^{n-t}\widehat{x}_2 + Cx_3^{n-t}\widehat{x}_3\right)\left(Ax_1^{n+t}\widehat{x}_1 + Bx_2^{n+t}\widehat{x}_2 + Cx_3^{n+t}\widehat{x}_3\right) \\
& \quad - \left(Ax_1^{n}\widehat{x}_1 + Bx_2^{n}\widehat{x}_2 + Cx_3^{n}\widehat{x}_3\right)^2 \\
& = ABx_1^n x_2^n \widehat{x}_1\widehat{x}_2 \left(\frac{{x_1}^t}{{x_2}^t} + \frac{{x_2}^t}{{x_1}^t} - 2\right) + ACx_1^n x_3^n \widehat{x}_1\widehat{x}_3 \left(\frac{{x_1}^t}{{x_3}^t} + \frac{{x_3}^t}{{x_1}^t} - 2\right) \\
& + BCx_2^n x_3^n \widehat{x}_2\widehat{x}_3 \left(\frac{{x_2}^t}{{x_3}^t} + \frac{{x_3}^t}{{x_2}^t} - 2\right) \\
& = AB(x_1x_2)^{n-t}\widehat{x}_1\widehat{x}_2\left(x_1^t - x_2^t\right)^2 + AC(x_1x_3)^{n-t}\widehat{x}_1\widehat{x}_3\left(x_1^2 - x_3^2\right)^2 \\
& + BC(x_2x_3)^{n-t}\widehat{x}_2\widehat{x}_3\left(x_2^t - x_3^2\right)^2 \\
& = ABl^{n-1}{x_3}^{1-n}\widehat{x}_1\widehat{x}_2(x_1 - x_2)^2 + ACl^{n-1}{x_2}^{1-n}\widehat{x}_1\widehat{x}_3(x_1 - x_3)^2 \\
& + BCl^{n-1}{x_1}^{1-n}\widehat{x}_2\widehat{x}_3(x_2 - x_3)^2 \\
& = l^{n-t}\left(AB{x_3}^{t-n}\widehat{x}_1\widehat{x}_2\left(x_1^t - x_2^t\right)^2 + AC{x_2}^{t-n}\widehat{x}_1\widehat{x}_3\left(x_1^t - x_3^t\right)^2\right. \\
& \left. + BC{x_1}^{t-n}\widehat{x}_2\widehat{x}_3\left(x_2^t - x_3^t\right)^2\right).
\end{aligned}
$$

So, the desired result is obtained □

If we take $t = 1$, we get the Cassini identity.

Theorem 2.9. *(d'Ocagne Identity) Let n and m be any integers. Then the following identity is true.*

$$
\begin{aligned}
& BGP_{(k,l),m+1} BGP_{(k,l),n} - BGP_{(k,l),m} BGP_{(k,l),n+1} \\
& = AB\widehat{x}_1\widehat{x}_2\left(x_1 - x_2\right)\left(x_1^m x_2^n - x_2^m x_1^n\right) + AC\widehat{x}_1\widehat{x}_3\left(x_1 - x_3\right)\left(x_1^m x_3^n - x_3^m x_1^n\right) \\
& + BC\widehat{x}_2\widehat{x}_3\left(x_2 - x_3\right)\left(x_2^m x_3^n - x_3^m x_2^n\right)
\end{aligned}
$$

Proof. We use the Binet formula of the bicomplex $(k, l)-$Gadovan numbers for the proof.

$$
\begin{aligned}
& BGP_{(k,l),m+1} BGP_{(k,l),n} - BGP_{(k,l),m} BGP_{(k,l),n+1} \\
& = \left(Ax_1^{m+1}\widehat{x}_1 + Bx_2^{m+1}\widehat{x}_2 + Cx_3^{m+1}\widehat{x}_3\right)\left(Ax_1^{n}\widehat{x}_1 + Bx_2^{n}\widehat{x}_2 + Cx_3^{n}\widehat{x}_3\right) \\
& \quad - \left(Ax_1^{m}\widehat{x}_1 + Bx_2^{m}\widehat{x}_2 + Cx_3^{m}\widehat{x}31\right)\left(Ax_1^{n+1}\widehat{x}_1 + Bx_2^{n+1}\widehat{x}_2 + Cx_3^{n+1}\widehat{x}_3\right) \\
& = ABx_1^{m+1}x_2^n\widehat{x}_1\widehat{x}_2 + ACx_1^{m+1}x_3^n\widehat{x}_1\widehat{x}_3 + BAx_2^{m+1}x_1^n\widehat{x}_2\widehat{x}_1 \\
& \quad + BCx_2^{m+1}x_3^n\widehat{x}_2\widehat{x}_3 + CAx_3^{m+1}x_1^n\widehat{x}_3\widehat{x}_1 + CBx_3^{m+1}x_2^n\widehat{x}_3\widehat{x}_2 \\
& \quad - ABx_1^{m}x_2^{n+1}\widehat{x}_1\widehat{x}_2 - ACx_1^{m}x_3^{n+1}\widehat{x}_1\widehat{x}_3 - BAx_2^{m}x_1^{n+1}\widehat{x}_2\widehat{x}_1 \\
& \quad - BCx_2^{m}x_3^{n+1}\widehat{x}_2\widehat{x}_3 - CAx_3^{m}x_1^{n+1}\widehat{x}_3\widehat{x}_1 - CBx_3^{m}x_2^{n+1}\widehat{x}_3\widehat{x}_2 \\
& = AB\widehat{x}_1\widehat{x}_2x_1^m x_2^n\left(x_1 - x_2\right) + AB\widehat{x}_1\widehat{x}_2x_2^m x_1^n\left(x_2 - x_1\right) + AC\widehat{x}_1\widehat{x}_3x_1^m x_3^n\left(x_1 - x_3\right) \\
& \quad + AC\widehat{x}_1\widehat{x}_3x_3^m x_1^n\left(x_3 - x_1\right) + BC\widehat{x}_1\widehat{x}_3x_2^m x_3^n\left(x_2 - x_3\right) + BC\widehat{x}_2\widehat{x}_3x_3^m x_2^n\left(x_3 - x_2\right) \\
& = AB\widehat{x}_1\widehat{x}_2\left(x_1 - x_2\right)\left(x_1^m x_2^n - x_2^m x_1^n\right) + AC\widehat{x}_1\widehat{x}_3\left(x_1 - x_3\right)\left(x_1^m x_3^n - x_3^m x_1^n\right) \\
& \quad + BC\widehat{x}_2\widehat{x}_3\left(x_2 - x_3\right)\left(x_2^m x_3^n - x_3^m x_2^n\right).
\end{aligned}
$$

So, the proof is complete. $\square$

Theorem 2.10. *(Honsberger Identity) Let n and m be any integers. Then the following identity is true.*

$$
\begin{aligned}
&BGP_{(k,l),m}BGP_{(k,l),n}+BGP_{(k,l),m+1}BGP_{(k,l),n+1}\\
&=A^2x_1^{m+n}\widehat{x}_1^2\left(1+x_1^2\right)+B^2x_2^{m+n}\widehat{x}_2^2\left(1+x_2^2\right)+C^2x_3^{m+n}\widehat{x}_3^2\left(1+x_3^2\right)\\
&+AB(\widehat{x}_1\widehat{x}_2)(l+x_3)l^m{x_3}^{-m-1}+AC(\widehat{x}_1\widehat{x}_3)(l+x_2)l^m{x_2}^{-m-1}\\
&+BC(\widehat{x}_2\widehat{x}_3)(l+x_1)l^m{x_1}^{-m-1}.
\end{aligned}
$$

Proof. We use the Binet formula of the bicomplex $(k,l)-$Gadovan numbers for the proof.

$$
\begin{aligned}
&BGP_{(k,l),m}BGP_{(k,l),n}+BGP_{(k,l),m+1}BGP_{(k,l),n+1}\\
&=\left(Ax_1^m\widehat{x}_1+Bx_2^m\widehat{x}_2+Cx_3^m\widehat{x}_3\right)\left(Ax_1^n\widehat{x}_1+Bx_2^n\widehat{x}_2+Cx_3^n\widehat{x}_3\right)\\
&\quad+\left(Ax_1^{m+1}\widehat{x}_1+Bx_2^{m+1}\widehat{x}_2+Cx_3^{m+1}\widehat{x}_3\right)\left(Ax_1^{n+1}\widehat{x}_1+Bx_2^{n+1}\widehat{x}_2+Cx_3^{n+1}\widehat{x}_3\right)\\
&=A^2x_1^{m+n}\widehat{x}_1^2+ABx_1^mx_2^n\widehat{x}_1\widehat{x}_2+ACx_1^mx_3^n\widehat{x}_1\widehat{x}_3+BAx_2^mx_1^n+B^2x_2^{m+n}\widehat{x}_2^2\\
&\quad+BCx_2^mx_3^n\widehat{x}_2\widehat{x}_3+CAx_3^mx_1^n\widehat{x}_1\widehat{x}_3+CBx_3^mx_2^n\widehat{x}_3\widehat{x}_2+C^2x_3^{m+n}\widehat{x}_3^2\\
&\quad+A^2x_1^{m+n+2}\widehat{x}_1^2+ABx_1^{m+1}x_2^{n+1}\widehat{x}_1\widehat{x}_2+ACx_1^{m+1}x_3^{n+1}\widehat{x}_1\widehat{x}_3+BAx_2^{m+1}x_1^{n+1}\widehat{x}_2\widehat{x}_1\\
&+B^2x_2^{m+n+2}\widehat{x}_2^2\\
&\quad+BCx_2^{m+1}x_3^{n+1}\widehat{x}_2\widehat{x}_3+CAx_3^{m+1}x_1^{n+1}\widehat{x}_3\widehat{x}_1+CBx_3^{m+1}x_2^{n+1}\widehat{x}_2\widehat{x}_3+C^2x_3^{m+n+2}\widehat{x}_3^2\\
&=A^2x_1^{m+n}\widehat{x}_1^2\left(1+x_1^2\right)+B^2x_2^{m+n}\widehat{x}_2^2\left(1+x_2^2\right)+C^2x_3^{m+n}\widehat{x}_3^2\left(1+x_3^2\right)\\
&\quad+AB\widehat{x}_1\widehat{x}_2\left(x_1^mx_2^n+x_2^mx_1^n+x_1^{m+1}x_2^{n+1}+x_2^{m+1}x_1^{n+1}\right)\\
&\quad+AC\widehat{x}_1\widehat{x}_3\left(x_1^mx_3^n+x_3^mx_1^n+x_1^{m+1}x_3^{n+1}+x_3^{m+1}x_1^{n+1}\right)\\
&\quad+BC\widehat{x}_2\widehat{x}_3\left(x_2^mx_3^n+x_3^mx_2^n+x_2^{m+1}x_3^{n+1}+x_3^{m+1}x_2^{n+1}\right)\\
&=AB\widehat{x}_1\widehat{x}_2(1+x_1x_2)(x_1x_2)^m(x_2^{n-m}+x_1^{n-m})\\
&\quad+AC\widehat{x}_1\widehat{x}_3(1+x_1x_3)(x_1x_3)^m(x_1^{n-m}+x_3^{n-m})\\
&\quad+BC\widehat{x}_2\widehat{x}_3(1+x_2x_3)(x_2x_3)^m(x_2^{n-m}+x_3^{n-m})+A^2x_1^{m+n}\widehat{x}_1^2\left(1+x_1^2\right)\\
&\quad+B^2x_2^{m+n}\widehat{x}_2^2\left(1+x_2^2\right)+C^2x_3^{m+n}\widehat{x}_3^2\left(1+x_3^2\right)\\
&=A^2x_1^{m+n}\widehat{x}_1^2\left(1+x_1^2\right)+B^2x_2^{m+n}\widehat{x}_2^2\left(1+x_2^2\right)+C^2x_3^{m+n}\widehat{x}_3^2\left(1+x_3^2\right)\\
&\quad+AB(\widehat{x}_1\widehat{x}_2)(l+x_3)l^m{x_3}^{-m-1}\\
&\quad+AC(\widehat{x}_1\widehat{x}_3)(l+x_2)l^m{x_2}^{-m-1}\\
&+BC(\widehat{x}_2\widehat{x}_3)(l+x_1)l^m{x_1}^{-m-1}.
\end{aligned}
$$

$\square$

Theorem 2.11. *(Vajda Identity) Let n and m be any integers. Then the following identity is true.*

$$\begin{aligned}
&BGP_{(k,l),n+m}BGP_{(k,l),n+r}-BGP_{(k,l),n}BGP_{(k,l),n+m+r}\\
&=ABx_1^nx_2^n\widehat{x}_1\widehat{x}_2\left(x_1^r-x_2^r\right)\left(x_2^m-x_1^m\right)+ACx_1^nx_3^n\widehat{x}_1\widehat{x}_3\left(x_1^r-x_3^r\right)\left(x_3^m-x_1^m\right)\\
&\quad+BCx_2^nx_3^n\widehat{x}_2\widehat{x}_3\left(x_2^r-x_3^r\right)\left(x_3^m-x_2^m\right).
\end{aligned}$$

Proof. We use the Binet formula of the bicomplex $(k,l)-$Gadovan numbers for the proof.

$$\begin{aligned}
&BGP_{(k,l),n+m}BGP_{(k,l),n+r}-BGP_{(k,l),n}BGP_{(k,l),n+m+r}\\
&=\left(Ax_1^{n+m}\widehat{x}_1+Bx_2^{n+m}\widehat{x}_2+Cx_3^{n+m}\widehat{x}_3\right)\left(Ax_1^{n+r}\widehat{x}_1+Bx_2^{n+r}\widehat{x}_2+Cx_3^{n+r}\widehat{x}_3\right)\\
&\quad-\left(Ax_1^n\widehat{x}_1+Bx_2^n\widehat{x}_2+Cx_3^n\widehat{x}_3\right)\left(Ax_1^{n+m+r}\widehat{x}_1+Bx_2^{n+m+r}\widehat{x}_2+Cx_3^{n+m+r}\widehat{x}_3\right)\\
&=ABx_1^{n+m}x_2^{n+r}\widehat{x}_1\widehat{x}_2+ACx_1^{n+m}x_3^{n+r}\widehat{x}_1\widehat{x}_3+BAx_2^{n+m}x_1^{n+r}\widehat{x}_2\widehat{x}_1\\
&\quad+BCx_2^{n+m}x_3^{n+r}\widehat{x}_2\widehat{x}_3+CAx_3^{n+m}x_1^{n+r}\widehat{x}_3\widehat{x}_1+CBx_3^{n+m}x_2^{n+r}\widehat{x}_3\widehat{x}_2\\
&\quad-ABx_1^nx_2^{n+m+r}\widehat{x}_1\widehat{x}_2-ACx_1^nx_3^{n+m+r}\widehat{x}_1\widehat{x}_3\\
&\quad-BAx_2^nx_1^{n+m+r}\widehat{x}_2\widehat{x}_1-BCx_2^nx_3^{n+m+r}\widehat{x}_2\widehat{x}_3\\
&\quad-CAx_3^nx_1^{n+m-r}\widehat{x}_3\widehat{x}_1-CBx_3^nx_2^{n+m+r}\widehat{x}_3\widehat{x}_2\\
&=ABx_1^nx_2^n\widehat{x}_1\widehat{x}_2\left(x_1^r-x_2^r\right)\left(x_2^m-x_1^m\right)\\
&\quad+ACx_1^nx_3^n\widehat{x}_1\widehat{x}_3\left(x_1^r-x_3^r\right)\left(x_3^m-x_1^m\right)\\
&\quad+BCx_2^nx_3^n\widehat{x}_2\widehat{x}_3\left(x_2^r-x_3^r\right)\left(x_3^m-x_2^m\right).
\end{aligned}$$

So, the proof is complete. □

3. Conclusion

We introduced the bicomplex $(k,l)-$Gadovan numbers and some identities of the bicomplex $(k,l)-$Gadovan numbers. We obtained relation to some numbers, Binet formula, binomial sums, the generating functions. We obtained Cassini, Catalan, Vajda, Honsberger and Dâ'ocagne identities, which are important identities related to number sequences.

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